AN EQUATION FOR CONTINUOUS CHAOS

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A prototype equation to the Lorenz model of turbulence contains just one (second-order) nonlinearity in one variable. The flow in state space allows for a "folded" Poincaré map (horseshoe map). Many more natural and artificial systems are governed by this type of equation.

Continuous chaos has, under the name of deterministic nonperiodic flow, been first described by E.N. Lorenz in a model of turbulence [1]. The same model has recently been found to apply to lasers as well, explaining the phenomenon of irregularly spiking lasers in this case [2]. The Lorenz equation consists of three coupled ordinary differential equations which contain two nonlinear terms (of second order, xz and xy):

$$\dot{x} = 10(y - x), \quad \dot{y} = x(28 - z) - y, \quad \dot{z} = xy - \tfrac{8}{3}z. \tag{1}$$

The flow of trajectories in state space (fig. 1) shows two unstable foci (spirals) suspended in an attracting surface each, and mutually connected in such a way that the outer portion of either spiral is "glued" toward the side of the other spiral, whereby the outermost parts of the first spiral map onto the more inner parts of the second, and vice versa. Unexpectedly, the qualitative behavior of eq. (1) is still insufficiently understood, mainly because the usual technique for analyzing oscillations – to find a (Poincaré) cross-section through the flow which is a (auto-) diffeomorphism [3] – is not applicable. A trick which exploits the inherent (though imperfect) symmetry between the two "leaves" of the flow (see fig. 1), so that in effect only a single leaf needs to be considered, has yet to be found.

Therefore, a simpler equation which directly generates a similar flow and forms only a single spiral may be of interest, even if this equation has, as a "model of a model", no longer an immediate physical interpretation. The proposed equation is:

$$\dot{x} = -(y + z), \quad \dot{y} = x + 0.2y, \quad \dot{z} = 0.2 + z(x - 5.7). \tag{2}$$

There is only a single nonlinear term (zx) now. The

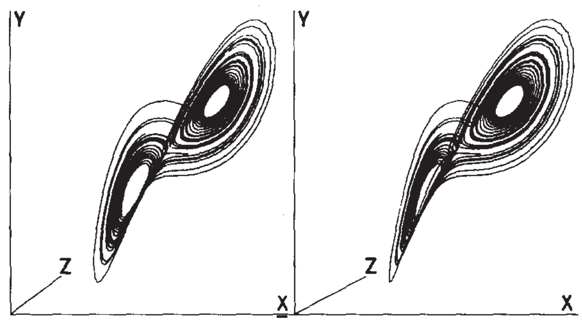

Fig. 1. Trajectories of the Lorenz model (eq. 1). Stereoscopic view. (Parallel projections; the left-hand picture is meant for the right eye and vice versa.) Numerical simulation on a HP9820A calculator with peripherals, using a standard Runge-Kutta-Merson integration routine (adapted by F. Göbber). Axes: -29 ... $+29$ for x and y, 0 ... 58 for z. Initial conditions assumed: $x(0) = 2.9$, $y(0) = -1.3$, $z(0) = 25$. Final values: $t_{\text{end}} = 31.668$, $x(\text{end}) = -4.451$, $y(\text{end}) = 2.3833$, $z(\text{end}) = 30.933$.

generated flow (fig. 2) is that of a (disk-embedded) *single* spiral. The outer portion returns, after an appropriate twist (so that the formation of a Möbius band is involved [4]), toward the side of the same spiral, with the outermost parts again facing the more central parts. The trajectorial convolute looks much like that on a single leaf of fig. 1. This time, however, a qualitative understanding of the "chaotic flow" (a term coined by Yorke for analogous discrete systems; see ref. [5] and below) is easier to obtain.

By drawing an unwinding spiral on a transparent sheet of paper, folding the sheet over, and gluing the outer part of the spiral onto the inner one, an analog to the flow of fig. 2 is obtained. When carefully following-up the prescribed course of a trajectory within

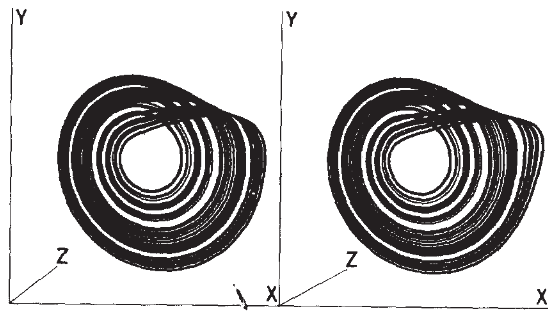

Fig. 2. Trajectorial flow of eq. (2). Stereoplot as in fig. 1. Axes: $-14 \ldots +14$ for x and y, $0 \ldots 28$ for z. Assumed initial conditions: $x(0) = 0$, $y(0) = -6.78$, $z(0) = 0.02$. Final values: $t_{end} = 339.249$, $x(end) = -7.8366$, $y(end) = -4.1803$, $z(end) = 0.014385$.

this "trap", one comes up with a picture very much like that of fig. 2. If one then varies the degree of overlap, it is apparent that nonperiodic behavior is obtained if and only if at least two successive increases of amplitude are possible for the outermost trajectory, after it has become the innermost trajectory. Most recently, a proof of this result has been described (under the suggestive title "period 3 implies chaos") for one-dimensional "cap-shaped" maps [5]. Such a map will indeed be found along any cross-section through the desired paper-sheet flow, if the reentry point through the cross-section is plotted as a function of the entry point. (The converse is also true: every cap-shaped map gives rise to a paper-sheet flow possessing this map as a Poincaré cross-section.)

Closer inspection of fig. 2 reveals, however, that the flow actually is *not* confined to a (folded) two-dimensional surface, but rather to a (folded) disk of finite width. Every cross-section through the flow is therefore two-dimensional (rather than one-dimensional). It assumes the form of a horseshoe between one transition and the next. This becomes evident if one follows the course of one (at first) rectangular cross-section as it is "stretched" and then "folded" before it is mapped back onto itself.

As it turns out, the properties of such "folded" diffeomorphisms, called horseshoe maps [3], are well-known in the theory of dynamical systems, and so is the fact that each of them can give rise to a three-dimensional "suspended" flow [3]. Only a simple (three-dimensional) example had been lacking so far [4]. Thus, the limit set is a so-called strange attractor [6] whose cross-section is a two-dimensional Cantor set; the flow is nonperiodic and structurally stable [6], even though all trajectories are unstable [1]. Thus, most of the results which have been conjectured about eq. (1) [1] turn out to be true for eq. (2). The simplicity (not to say: triviality) of eq. (2) has the additional asset that some further results that one would like to know about strange attractors in general (basin structure; emergence through hard and soft bifurcation; behavior of the monostable variant; behavior under time reversal) may be easier to obtain with this equation.

Eq. (2) incidentally illustrates a more general principle for the generation of "spiral type" chaos [7]: combining a two-variable oscillator (in this case x and y) with a switching-type subsystem (z) in such a way that the latter is being switched by the first while the flow of the first is dependent on the switching state of the latter. Eq. (2) has in fact been derived from a more complicated equation for which this "building-block principle" has been shown to apply strictly [4]. The named design principle not only enables the construction of an unlimited number of artificial chaotic systems, but at the same time can be used as a guideline for the identification of further natural systems showing the same behavior (by suggesting to probe into their parameter space). The field of possible applications of equations of the *type* of eq. (2) thus ranges from astrophysics, via chemistry and biology, to economics [7].

To conlude, continuous chaos is "stangely attractive" as a physical phenomenon (cf. [8]).

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